GIRRAWEEN HIGH SCHOOL

HALF-YEARLY EXAMINATION

YEAR 12

2015

MATHEMATICS

Time allowed – Two hours

(Plus 5 minutes reading time)

DIRECTIONS TO CANDIDATES

- Attempt all questions.
- Circle the best response for the questions in Part A.
- Start each question in Part B on a new page.
- All necessary working must be shown.
- Marks may be deducted for careless or badly arranged work.

PART A (10 marks)

Question 1

0.0398 to 2 significant figures is:

(a) 0.03 (b) 0.04 (c) 0.039 (d) 0.040

Question 2

The solution to $35 - 3x > 19 - x$ is:

(a) $x < 8$ (b) $x > 8$ (c) $x \leq 8$ (d) $x \geq 8$

Question 3

If α and β are the roots to the equation $x^2 + 4x + 1 = 0$, then the value of $\alpha + \beta$ is:

(a) $\frac{1}{4}$ (b) 4 (c) $-\frac{1}{4}$ (d) -4

Question 4

The perpendicular distance from the point (2, 3) and the line $6x + 8y - 5 = 0$ is:

(a) $\frac{31}{10}$ (b) $\frac{31}{100}$ (c) $\frac{17}{10}$ (d) $\frac{17}{100}$

Question 5

Evaluate $\sum_{x=1}^{4} x^x$

(a) 1 (b) 4 (c) 256 (d) 288

Question 6

The focus of $x^2 = -16y$ is:

(a) (0, 4) (b) (4, 0) (c) (0, -4) (d) (-4, 0)

Question 7

Evaluate $\lim_{x \to 3} \frac{x^2 - 9}{x - 3}$

(a) 0 (b) $\frac{1}{3}$ (c) 3 (d) 6

Question 8

The second derivative of $f(x) = x^2 + \frac{1}{x}$

(a) $2x - \frac{2}{x^2}$ (b) $2 - \frac{1}{x^3}$ (c) $2 + \frac{2}{x^3}$ (d) $2x + \frac{2}{x^2}$

Question 9

The solution to $|2x - 1| \geq 3$

(a) $x \geq 2$, $x \leq -1$ (b) $x \leq 2$, $x \geq -1$ (c) $x \geq -2$, $x \leq 1$ (d) $x \leq -2$, $x \geq 1$

Question 10

Which definite integral represents the area bounded by the curve $y = 4 - x^2$ and the x axis?

(a) $\int_0^2 (4 - x^2)dx$ (b) $\int_{-2}^0 (4 - x^2)dx$ (c) $\int_{-2}^2 (4 - x^2)dx$ (d) $\int_{-4}^4 (4 - x^2)dx$

PART B

Question 11 (17 marks)

(a) Differentiate the following. Simplify the answer if necessary.

(i) $y = x^6 + 2\sqrt{x}$ **(1)**

(ii) $y = 2e^{4x} + e^{-x}$ **(2)**

(iii) $y = e^{2x}(e^x - e^{-x})$ **(2)**

(iv) $y = x^2 e^{x^3}$ **(3)**

(v) $y = \dfrac{e^{4x}}{x-1}$ **(3)**

(b) Simplify $\log_6(16) + \log_6(81)$ **(3)**

(c) Use the trapezoidal rule with 5 function values to give an

approximation for $\int_3^7 (x^2 + 3x)dx$. **(3)**

Question 12 (15 marks)

(a) *P (2, 3), Q (6, -1)* and *R (-4, -5)* are the vertices of ΔPQR . M is the midpoint of *PQ* and *N* is the midpoint of *PR*.

(i) Draw a diagram showing this information. **(1)**

(ii) Find the co-ordinates of *M* and *N*. **(2)**

(ii) Show that $MN // QR$. **(2)**

(iii) Calculate the length of *QR*. **(2)**

(iv) Show that the length of *MN* is half the length of *QR*. **(2)**

(b) Evaluate $\int_0^1 e^{-\frac{x}{2}} dx$ using Simpson's rule with 3 function values, giving your answer as an exact value. **(3)**

(c) Solve for a, $\log_2 a - \log_2 3 = \log_2(a+2) - \log_2(a-2)$ **(3)**

Question 13 (16 marks)

(a) Consider the curve given by $y = 1 + 3x - x^3$, for $-2 \leq x \leq 3$.

(i) Find the stationary points and determine their nature. **(4)**

(ii) Find the point of inflexion. **(2)**

(iii) Sketch the curve for $-2 \leq x \leq 3$. **(3)**

(iv) What is the minimum value of y for $-2 \leq x \leq 3$. **(1)**

(b) A box contains 8 red and 11 green marbles. Arnav randomly selects three marbles one at a time and without replacement. What is the probability that he selects green, red then green in that order? **(2)**

(c) In a class of 24 students, 3 play no sport, 14 play cricket and 12 play tennis.

(i) Construct a Venn Diagram showing this information. **(1)**

(ii) If a student is selected at random, find the probability that:

(α) He or she plays tennis? **(1)**

(β) He or she plays both tennis and cricket? **(1)**

(γ) He or she doesn't play cricket? **(1)**

Examination continues on the next page

Question 14 (20 marks)

(a) Find the following integrals.

(i) $\int 5x^3 - 2x + \sqrt{x}\ dx$ **(2)**

(ii) $\int e^x + e^{-3x} dx$ **(2)**

(b) (i) Differentiate $y = e^{x^2}$ **(2)**

(ii) Hence, find $\int_0^2 2xe^{x^2} dx$ **(2)**

(c) Find the sum of the first twenty terms of an arithmetic series, given that the tenth term is 39 and the sum of the first ten terms is 165. **(4)**

(d) Find the values of x for which the series $1 + (x-3)^2 + (x-3)^4 + ...$ has a limiting sum. **(4)**

(e) Cans of fruit in a supermarket display are stacked so that there are 3 cans in the top row, 5 in the next row, 7 in the next row and so on. If there are 10 rows in the display, find:

(i) The number of cans in the bottom row **(2)**

(ii) The total number of cans in the display **(2)**

Question 15 (15 marks)

(a) A company finds that the function $f(x) = x^3 - 96x^2 + 2880x$ provides a good approximation for their profit $f(x)$ in dollars, where x is the advertising expenditure in thousands of dollars.

(i) What expenditure of advertising would produce the maximum profit? **(4)**

(ii) What is the maximum profit? **(2)**

(b) Find the value of k if $y = e^{kx}$ is a solution of $y'' - y' - 12y = 0$. **(4)**

(c) Sketch the graphs $y = e^{-x}$, $y = x + 1$ and the line $x = 2$. Find the area of the region bounded by all three curves. **(5)**

Question 16 (14 marks)

(a) A piece of wire 1 m long is cut into two parts. Each part is bent to form a circle. If one of the pieces of wire is $x\text{ cm}$ long:

(i) Show that the length in centimetres of the radii of the two circles are

$$\frac{x}{2\pi} \text{ and } \frac{100-x}{2\pi}.$$ **(2)**

(ii) Show that the sum of the areas of the circles is

$$\frac{1}{2\pi}(x^2-100x+5000)\text{ cm}^2.$$ **(4)**

(iii) Find the lengths of each piece of wire so that the sum of their areas is least. **(4)**

(b) A hemispherical bowl of radius a units is filled with water to a depth of $\frac{a}{2}$ units. Show that the volume of water is $\frac{5a^3\pi}{24}$ cubic units. **(4)**

END OF EXAMINATION ☺

YR 12 2U Half Yearly 2015

PART A Multiple Choice

1. D 2. A 3. D 4. A 5. D
6. C 7. B 8. C 9. A 10. C

PART B - Question 11

a) i) $y = x^6 + 2\sqrt{x}$

$y' = 6x^5 + \dfrac{1}{\sqrt{x}}$ (1)

ii) $y = e^{2x}(e^x - e^{-x})$

$= e^{3x} - e^x$

$y' = 3e^{3x} - e^x$ (2)

v) $y = x^2 e^{x^3}$

$u = x^2$
$u' = 2x$
$v = e^{x^3}$
$v' = 3x^2 e^{x^3}$

$y' = e^{x^3}(2x) + x^2(3x^2 e^{x^3})$

$= 2xe^{x^3} + 3x^4 e^{x^3}$

$= xe^{x^3}(2 + 3x^3)$ (3)

vi) $y = \dfrac{e^{4x}}{x-1}$

$u = e^{4x}$
$u' = 4e^{4x}$
$v = x-1$
$v' = 1.$

$y' = \dfrac{(x-1)(4e^{4x}) - e^{4x}(1)}{(x-1)^2}$

$= \dfrac{4xe^{4x} - 4e^{4x} - e^{4x}}{(x-1)^2}$ (3)

$= \dfrac{e^{4x}(4x-5)}{(x-1)^2}$

b) $\log_6 16 + \log_6 81$

$= \log_6 2^4 + \log_6 3^4$

$= 4(\log_6 2 + \log_6 3)$ (3)

$= 4(\log_6 6)$

$= 4.$

c) i) $\displaystyle\int_3^7 (x^2 + 3x)\,dx$

$h = \dfrac{7-3}{4} = 1.$

x	3	4	5	6	7
$f(x)$	18	28	40	54	70

$\displaystyle\int_3^7 (x^2 + 3x) \doteq$

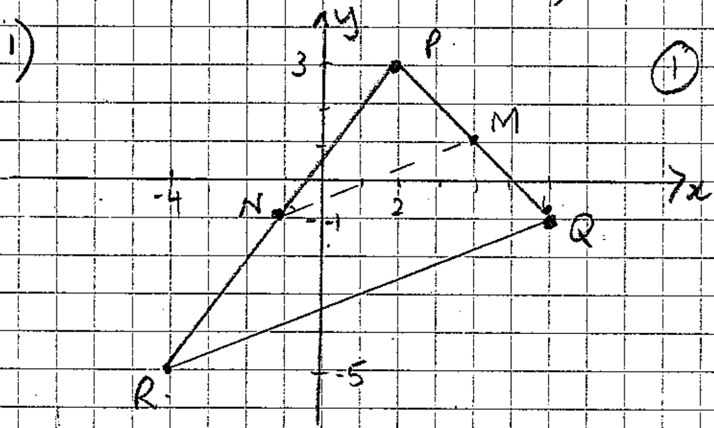
$\dfrac{1}{2}\left[18 + 70 + 2(28 + 40 + 54)\right]$ (3)

$= \dfrac{332}{2} = 166$

Question 12

a) $P(2,3)$ $Q(6,-1)$ $R(-4,-5)$

i)



(1)

ii) $M_{PQ} = \left(\frac{2+6}{2}, \frac{3-1}{2}\right) = (4, 1)$

$M_{PR} = \left(\frac{2-4}{2}, \frac{3-5}{2}\right) = (-1, -1)$ (2)

$\therefore M(4,1)$ $N(-1,-1)$

iii) $MN \parallel QR$

$m_{MN} = \frac{1+1}{4+1} = \frac{2}{5}$ $m_{QR} = \frac{-1+5}{6+4} = \frac{2}{5}$

$\therefore m_{MN} = m_{QR}$ $\therefore MN \parallel QR$ (2)

iv) $d_{QR} = \sqrt{(6+4)^2 + (-1+5)^2}$

$= \sqrt{10^2 + 4^2}$ (2)

$= \sqrt{116}$

v) $MN^2 = (x_2 - x_1)^2 + (y_2 - y_1)^2$

$= (4+1)^2 + (1+1)^2$

$= 5^2 + 2^2$ (2)

$MN^2 = 29$

$MN = \sqrt{29}$ units

now $2 \times \sqrt{29} = \sqrt{4} \times \sqrt{29}$

$= \sqrt{116}$ units

$\therefore 2 \times MN = QR$

ie $MN = \frac{1}{2} QR$.

b) $\int_0^1 e^{-\frac{x}{2}} dx$

$h = \frac{1-0}{2} = \frac{1}{2}$.

x	0	$\frac{1}{2}$	1
$f(x)$	1	$\frac{1}{e^{\frac{1}{4}}}$	$\frac{1}{\sqrt{e}}$

$\int_0^1 e^{-\frac{x}{2}} dx \doteqdot$

$\frac{\frac{1}{2}}{3}\left[1 + \frac{1}{\sqrt{e}} + 4\left(\frac{1}{e^{\frac{1}{4}}}\right)\right]$

$= \frac{1}{6}\left(1 + \frac{1}{\sqrt{e}} + \frac{4}{e^{\frac{1}{4}}}\right)$ (3)

c) $\log_2 a - \log_2 3 = \log_2(a+2) - \log_2(a-2)$

$\log_2 \frac{a}{3} = \log_2\left(\frac{a+2}{a-2}\right)$

$\therefore \frac{a}{3} = \frac{a+2}{a-2}$

$a(a-2) = 3(a+2)$ (3)

$a^2 - 2a = 3a + 6$

$a^2 - 5a - 6 = 0$

$(a-6)(a+1) = 0$

$\therefore a = 6$ $a = -1$

$\therefore a = 6$ as $a > 0$

Question 13.

a) $y = 1 + 3x - x^3, \quad -2 \le x \le 3$

i) $y' = 3 - 3x^2$ $\qquad y' = 0$

$3 - 3x^2 = 0$

$3x^2 = 3$

$x^2 = 1$

$\therefore x = \pm 1$

When $x = 1$, $y = 3$

When $x = -1$, $y = -1$

$\therefore$ Stationary pts at $(1, 3)$ and $(-1, -1)$

$y'' = -6x$

When $x = 1$ $\quad y'' = -6 \quad y'' < 0 \quad \therefore$ max.

When $x = -1$ $\quad y'' = 6 \quad y'' > 0 \quad \therefore$ min. (4)

$(1, 3)$ is a max. turning pt.

$(-1, -1)$ is a min. turning pt.

ii) When $y'' = 0$

$-6x = 0$

$\therefore x = 0$

When $x = 0$, $y = 1$. possible point of inflexion (2)

Test.

x	-1	0	1
y''	6	0	-6

change in concavity

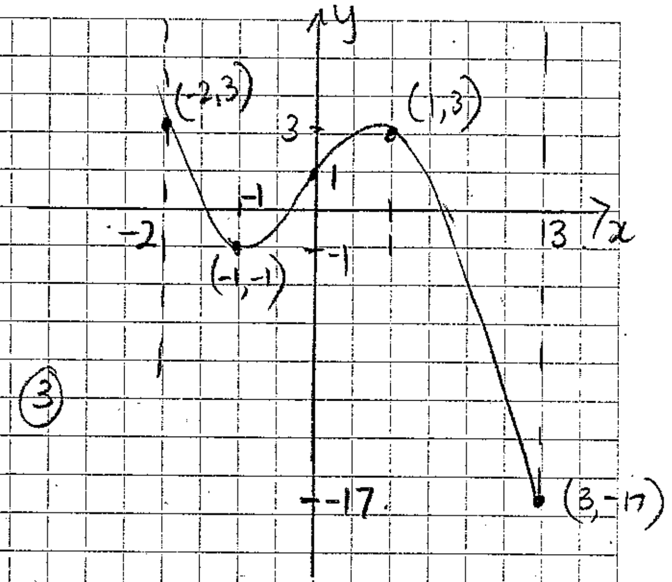
$\therefore (0, 1)$ is a point of inflexion.

iii) When $x = -2$ $\qquad x = 3$

$y = 1 + 3(-2) - (-2)^3 \qquad y = 1 + 3(3) - (3)^3$

$= 1 - 6 + 8 \qquad = 1 + 9 - 27$

$= 3 \qquad = -17$



(3)

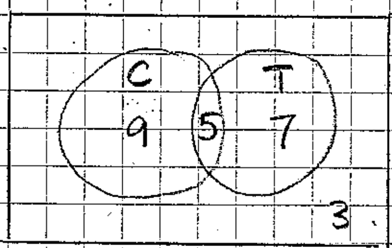
iv) minimum value y is -17. (1)

b) 8 R 11 G

$P(GRG) = \left(\frac{11}{19} \times \frac{8}{18} \times \frac{10}{17}\right)$ (2)

$= \frac{880}{5814} = \frac{440}{2907}$

c) i)



(1)

ii) $P(\text{plays tennis}) = \frac{12}{24} = \frac{1}{2}$ (1)

iii) $P(\text{plays both}) = \frac{5}{24}$ (1)

iv) $P(\text{Don't play cricket})$

$= \frac{10}{24} = \frac{5}{12}$ (1)

Question 14.

a) i) $\int 5x^3 - 2x + \sqrt{x}\, dx$ ②

$= \frac{5x^4}{4} - x^2 + \frac{2}{3}\sqrt{x^3} + c$

ii) $\int e^x + e^{-3x}\, dx$ ②

$= e^x - \frac{1}{3}e^{-3x} + c$

b) i) $y = e^{x^2}$

$y' = 2xe^{x^2}$ ①

ii) $\int_0^2 2xe^{x^2} dx$

$= \left[e^{x^2}\right]_0^2$ ③

$= e^4 - e^0$

$= e^4 - 1$

c) $T_{10} = 39 \quad S_{10} = 165$

$T_{10} = a + 9d \quad S_{10} = \frac{10}{2}[2a + 9d]$

$a + 9d = 39$ ① $\quad 5(2a + 9d) = 165$

$2a + 9d = 33$ ②

$a = -6$

$\therefore -6 + 9d = 39$

$9d = 45$ ④

$d = 5$

$\therefore S_{20} = \frac{20}{2}\left[2(-6) + 19(5)\right]$

$= 10[-12 + 95)$

$= 830$

d) $r = \frac{(x-3)^2}{1} = \frac{(x-3)^4}{(x-3)^2} = (x-3)^2$

$|r| < 1 \quad \therefore$ A geometric series

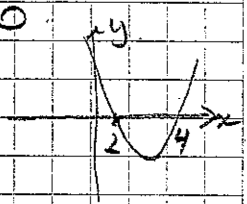
$(x-3)^2 < 1$

$x^2 - 6x + 9 < 1$ ④

$x^2 - 6x + 8 < 0$

$(x-4)(x-2) < 0$

$x = 4, 2$



$\therefore 2 < x < 4$

e) $a = 3 \quad d = 2$

i) $T_{10} = 3 + 9(2)$ ②

$= 3 + 18 = 21$

$\therefore$ There are 21 cars in the bottom row.

ii) $S_{10} = \frac{10}{2}\left[2(3) + 9(2)\right]$

$= 5[6 + 18]$ ②

$= 120$

$\therefore$ There are 120 cars on display.

Question 15.

a) $f(x) = x^3 - 96x^2 + 2880x$

i) $f'(x) = 3x^2 - 192x + 2880$

$f'(x) = 0$

$3x^2 - 192x + 2880 = 0$

$x^2 - 64x + 960 = 0$

$(x-40)(x-24) = 0$ (4)

$\therefore x = 40, 24$

$f''(x) = 6x - 192$

When $x = 40$ $f''(x) = 6(40) - 192$
$= 240 - 192$
$= 48$

$f''(x) > 0 \therefore$ min.

When $x = 24$ $f''(x) = 6(24) - 192$
$= 144 - 192$
$= -48$

$f''(x) < 0 \therefore$ max.

$\therefore$ \$24 would produce the maximum profit.

ii) $f(24) = 24^3 - 96(24)^2 + 2880(24)$

$= 27648$ (2)

$\therefore$ The maximum profit is \$27648.

b) $y = e^{kx}$ $y'' - y' - 12y = 0$

$y' = ke^{kx}$

$y'' = k^2e^{kx}$

$\therefore y'' - y' - 12y = k^2e^{kx} - ke^{kx} - 12(e^{kx})$

$= e^{kx}(k^2 - k - 12)$

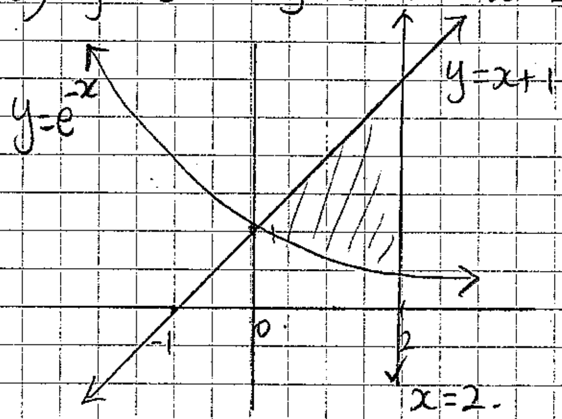
$\therefore e^{kx}(k^2 - k - 12) = 0$

$k^2 - k - 12 = 0$ (4)

$(k-4)(k+3) = 0$

$\therefore k = 4, -3$

c) $y = e^{-x}$ $y = x+1$ $x = 2$.



$A = \int_0^2 (x+1) - e^{-x}\, dx$ (5)

$= \left[\frac{x^2}{2} + x + e^{-x} \right]_0^2$

$= \left(\frac{2^2}{2} + 2 + e^{-2} \right) - \left(\frac{0^2}{2} + 0 + e^0 \right)$

$= \left(2 + 2 + \frac{1}{e^2} \right) - 1$

$= 3 + \frac{1}{e^2}$ units^2

Question 16

a) i) x and $100-x$ are the lengths of the two pieces of wire

Circle$_1$ $2\pi r = x$ $\therefore r = \dfrac{x}{2\pi}$

Circle$_2$ $2\pi r = 100 - x$ $\therefore r = \dfrac{100-x}{2\pi}$ (2)

ii) $A_1 = \pi r^2 = \pi\left(\dfrac{x}{2\pi}\right)^2 = \dfrac{x^2}{4\pi}$ cm^2

$A_2 = \pi r^2 = \pi\left(\dfrac{100-x}{2\pi}\right)^2 = \dfrac{(100-x)^2}{4\pi}$ cm^2

$$A_1 + A_2 = \frac{x^2}{4\pi} + \frac{(100-x)^2}{4\pi}$$ (4)

$$= \frac{x^2 + (10000 - 200x + x^2)}{4\pi}$$

$$= \frac{2x^2 + 10000 - 200x}{4\pi}$$

$$= \frac{1}{2\pi}(x^2 - 100x + 5000) \text{ cm}^2$$

iii) $\dfrac{d}{dx}\left[\dfrac{1}{2\pi}(x^2 - 100x + 5000)\right]$

$$\frac{dA}{dx} = \frac{1}{2\pi}(2x - 100)$$

$$= \frac{1}{\pi}x - \frac{50}{\pi}$$

When $\dfrac{dA}{dx} = 0$

$$\frac{x}{\pi} - \frac{50}{\pi} = 0$$

$$x = 50$$

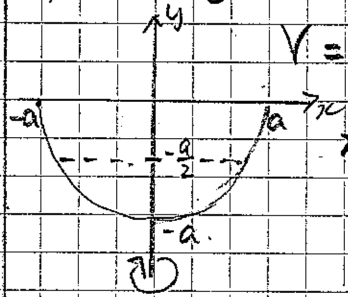
$\dfrac{d^2A}{dx^2} = \dfrac{1}{\pi}$ $\therefore$ always positive

$\therefore$ Only a minimum.

When $x = 50$ the sum of areas is least. (4)

$\therefore$ Each piece of wire is 50cm.

b) $x^2 + y^2 = a^2$ $\therefore x^2 = a^2 - y^2$



$$V = \pi\int_{-a}^{-\frac{a}{2}} a^2 - y^2\, dy$$

$$= \pi\left[a^2y - \frac{y^3}{3}\right]_{-a}^{-\frac{a}{2}}$$

$$= \pi\left[\left(a^2\left(-\frac{a}{2}\right) - \frac{\left(-\frac{a}{2}\right)^3}{3}\right) - \left(a^2(-a) - \frac{(-a)^3}{3}\right)\right]$$

$$= \pi\left[\left(-\frac{a^3}{2} + \frac{a^3}{24}\right) - \left(-a^3 + \frac{a^3}{3}\right)\right]$$

$$= \pi\left[\left(-\frac{11a^3}{24}\right) - \left(-\frac{2a^3}{3}\right)\right]$$ (4)

$$= \pi\left[-\frac{11a^3}{24} + \frac{16a^3}{24}\right]$$

$$= \frac{5a^3\pi}{24} \text{ units}^3$$